Some New Families of Total Vertex Product Cordial Labeling Of Graphs

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Abstract:- I.Cahit introduced cordial graphs as a weaker version of graceful and harmonious graphs. The total product cordial labeling is a variant of cordial labeling. In this paper we introduce a vertex analogue product cordial labeling as a variant of total product cordial labeling and name it as total vertex product cordial labeling. Finally, we investigate total vertex product cordial labeling for many families of graphs, namely fan graph, wheel graph, helm graph, double Star graph, cycle and fully binary tree graph.

Keywords- Graph Labeling, Cordial Graphs, Cordial Labeling, Product Cordial Labeling, Vertex Total Cordial Labeling, Total Vertex Product Cordial Labeling.

1 Introduction

All graphs mentioned in this paper are finite, simple and undirected. We follow the basic notations and terminology of graph labeling as in [1, 13] and as in [2].Over the last few decades the subject of graph labeling has been growing in popularity. Indeed, J.A. Gallian has compiled a periodically updates survey of many kind of labelings and so many results, obtained from well over a thousand referenced research articles [3, 10]. When one speaks of a labeling of a graph G, we typically refer to a mapping that carries graph elements to the numbers (usually to the integers). The most common choice of domain is the set of all vertices (vertex labeling), the edge set alone (edge labeling), or the set of all vertices and edges (total labeling). Other domains are also possible as Acharya introduced a new type of graph labeling in which domain is the collection of subsets of a set. First evidence of using assignment of subsets of a given set to the arcs of a diagraph appears in a seminal paper by Peay [15]. For all the terms and definitions related to set labeling we refers to [17, 18].

In most cases it is very interesting to consider the sum of all labels associated with a graph label. This will be called the weight of that element. As in a book of Wallis [16] the weight of a vertex under total labeling is

$$wt(x) = \lambda(x) + \sum_{xy\in E} \lambda(xy),$$

while

$$wt(xy) = \lambda(x) + \lambda(xy) + \lambda(y)$$

is called the weight of a vertex under total labeling.

The origin of the study of graph labeling as a major area of graph theory can be traced to a research paper by Rosa [4].In graph labeling magic and antimagic labelings have attracted the most interest and also have given rise to other labelings (super magic labeling, graceful labeling, harmonious labeling, cordial labeling etc). Among the magic and antimagic graphs cordial graphs are weaker version of graceful and harmonious graphs, so we will give brief summary of these two types of graphs. Among the labelings Rosa introduced a vertex labeling that was renamed as graceful labeling by Golomb [5].Let G be a graph of order p and size q, then G is said to be graceful if there exist a one-one function $f : V(G) \rightarrow \{0, 1, 2, \ldots\ldots q\}$ such that the induced function $f^*: E(G) \rightarrow \{1, 2, \ldots\ldots, q\}$ defined by $f^* : (uv) = |f(u)-f(v)|$ for each edge e = uv is a bijection then f is said to be graceful labeling of G. In order for a graph to

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posses graceful labeling, it is necessary that $p \geq q$-1.A graph possessing a graceful labeling is referred as graceful graph. A vast amount of literature is available on graceful labeling. As above we have mentioned that for a graph to posses graceful labeling, it is necessary that $p \geq q$-1, but every connected graph satisfying the above condition need not be graceful. For example C_5 and K_5 .Graham and Sloane [6] introduced harmonious labeling during their study on modular version of additive bases problems stemming from error correcting codes. Let f be a one-one function defined by $f : V(G) \rightarrow \{0, 1, 2, ...q\}$such that the induced function: $E(G) \rightarrow \{1, 2, ..., q\}$ defined by $f^* : (uv) = |f(u)+f(v)|$ for each edge e = uv is a bijection then f is said to be harmonious labeling of G and the graph G having such labeling is known a harmonious graph. Cordial labeling is a variation of both graceful and harmonious labeling.

First, we will give brief summary of definitions which are useful for investigations in this paper.

Definition: A closed path, consist a sequence of vertices starting and ended at the same vertex, with each two consecutive vertices in the sequence adjacent to each other in the graph.

Definition: The wheel Wn is defined to be the join C_n $+K_1$. The vertex of K_1 is known as apex vertex and vertices of C_n are known as rim vertices.

which each internal vertex has exactly two children.

Definition: The fan graph fn is defined as $K_1 + P_n$, where P_n is the path on n vertices.

Definition: A tree on n vertices is called a double star denoted by $S_{a, b}$ if it has exactly two vertices that are not leaves, one of degree a and the another of degree b, where

Definition: An ordered rooted tree is a binary tree in which each internal vertex has atmost two children.

Definition: A fully binary tree is a binary tree

2 Vertex Product cordial Labeling of Graphs

Cordial graph labelings were introduced by Cahit in 1987 as a weekend version of graceful and harmonious labelings. A graph is called cordial if it is possible to label its vertices with 0′s and 1's so that when the edges are labeled with the difference of the labels at their endpoints, the number of vertices (edges) labeled with one's and zero's differ at most by one. This labeling was further studied in, and motivates us to introduce a special

n = a+b. An example of a double star graph $S_{6, 7}$ is given in Figure (i)

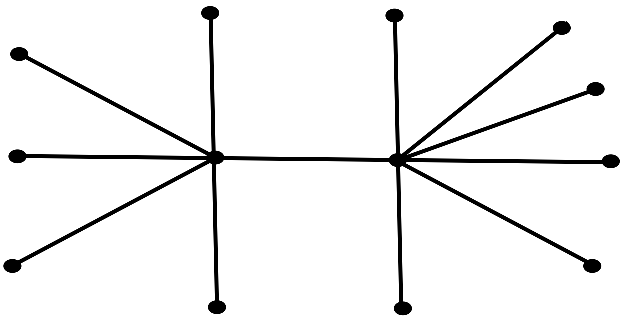


Figure (i)

Definition: The helm graph H_n is the graph obtained from an n- wheel graph by adjoining a pendant edge at each node of the cycle. It contains three types of vertices : n apex of degree four, an apex of degree n and n pendant vertices. Examples of helm graph H_4 and H_6 are given in Figure (ii) and Figure (iii)

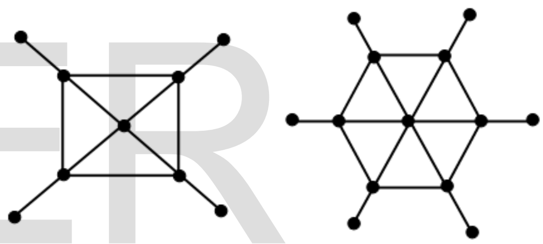


Figure (ii) and Figure (iii)

type of cordial labeling called vertex product cordial labeling of graphs. First, we will give brief summary of definitions which are useful for the present investigations.

Definition: Let G be a graph, a vertex labeling function $f : V(G) \rightarrow \{0,1\}$ induces an edge labeling function $f^* : E \rightarrow \{0, 1\}$ defined as $f^*(uv) = |f(u)- f(v)|$.Let $v_f(i)$ be the number of vertices of G having label i under f and e_{f*} (i) be the number of edges of G having label i under f* for i =0,1.The function f is called cordial labeling of G if $|v_f(1)-v_f(0)| \leq 1$ and $|e_{f*}(1)- e_{f*}(0)| \leq 1$. A graph is called cordial if admits cordial labeling. Cahit proved many results on cordial labeling [7].

Definition: Let G be a graph, an edge labeling function $f^*:E(G) \rightarrow \{0, 1\}$ induces a vertex labeling function $f: V(G) \rightarrow \{0, 1\}$defined as $f(u)=\{\Sigma f^*(uv) : uv \in E(G)\}(\text{mod } 2)$.The function is called E-cordial labeling of if $|v_f(1)-v_f(0)| \leq 1$ and $|e_{f*}(1)- e_{f*}(0)| \leq 1$. A graph is called E- cordial if it admits E-cordial labeling.

Definition: Let G be a graph, an edge labeling function f* : E(G)→{0, 1}induces a vertex labeling function f:V(G) →{ 0, 1 }defined as f(u)=Σ f* (uv) : uv∈ E(G)}(mod 2).The function f* is called a total edge cordial labeling of G if $|(e_{f*}(1)+v_f(1))-(e_{f*}(0)+v_f(0))|\le 1$. A graph is called total edge cordial if it admits total edge cordial labeling [9, 11, 12].

From the study of different type of cordial labelings, we introduce a new special type of cordial labeling called total vertex product cordial labeling as follows: If G is a graph, then a vertex labeling function f : V(G) →{ 0, 1 }induces an edge labeling function f*: E (G) →{ 0, 1 } defined as f*(uv)=f(u)f(v) then the function f is said to be total vertex product cordial labeling of G if $|(v_f(1)+ e_{f*}(1))-(v_f(0)+ e_{f*}(0))|\le 1$. A graph with a total vertex product cordial labeling is called a total vertex product cordial graph.

3 TOTAL VERTEX PRODUCT CORDIAL LABELING

It is easy to see that total vertex product cordial labeling of a graph is defined only for some connected graphs and for some graphs containing isolated vertex. Vertex product cordial labeling for many graphs having some another conditions can also be defined which we have not discussed in this paper. In this section total vertex product cordial labeling for wheel graph, double star graph, fan graph, helm graph, closed cycle and fully binary tree is defined. The proofs in this section as follow :

Theorem 3.1 The wheel graph Wn is total vertex product cordial graph.

Proof. Let $v_1,v_2,v_3\ldots v_n$ be the rim vertices and v be the apex vertex of wheel Wn. Let f: V (G) → {0, 1} defined by

Case (i): when n is even.

f(v)=1,

$$f(v_i)=\begin{cases}1 & \text{for } 1\le i\le \frac{n}{2},\\ 0 & \text{for } \left(\frac{n}{2}+1\right)\le i\le n.\end{cases}$$

In view of the above defined vertex labeling we have $|v_f(1)+e_{f*}(1)|=\frac{3n}{2}$and $|v_f(0)+e_{f*}(0)|=\frac{3n}{2}+1$. Thus in this case we have $|(v_f(1)+ e_{f*}(1))-(v_f(0)+ e_{f*}(0))|\le 1$.

Case (ii): when n is odd

f(v)=1,

$$f(v_i)=\begin{cases}1 & \text{for } 1\le i\le \frac{n}{2},\\ 0 & \text{for } \left(\frac{n}{2}+1\right)\le i\le n-2,\\ 1 & \text{for } i=n-1,\\ 0 & \text{for } i=n.\end{cases}$$

In view of the above defined vertex labeling we have $|v_f(1)+ e_{f*}(1)|=\frac{3n+1}{2}$and $|v_f(0)+ e_{f*}(0)|=\frac{3n+1}{2}$. Thus in this case we have $|(v_f(1)+e_{f*}(1))-(v_f(0)+ e_{f*}(0))|\le 1$.

Hence, the wheel graph Wn is total vertex product cordial graph.

Remark 1 : Function f^* in above theorem and in all other theorems represents an edge labeled function as defined in the definition of total vertex product cordial labeling.

Example 1. The wheel graphsW_6 and W_5 with its total vertex product cordial labeling are shown in Figure (iv) and Figure (v)

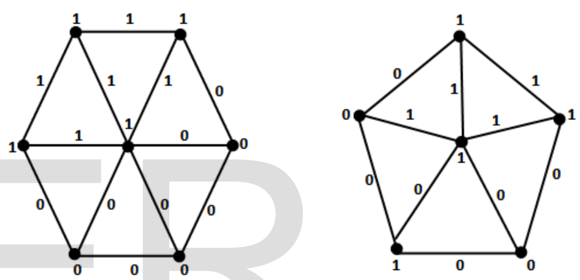


Figure (iv) and Figure (v)

Theorem 3.2 The double star graph $S_{m,n}$ is total vertex product cordial graph.

Proof. Let $S_{m,n}$ be a double star graph with u and v common vertices and $u_1, u_2, \ldots, u_{m-1}, v_1, v_2, \ldots, v_{n-1}$ be the other vertices. To define f: V(G) →{ 0, 1 } we consider the following cases.

Case (i):if m=n.

f(u)=1,

$f(u_i)=1$ for i=1, 2, ..., m-1,

f(v)=0,

$f(v_i)=0$ for i=1, 2, ..., n-1 .

In this case we have $|v_f(1)+ e_{f*}(1)|$ =2m -1 and $|v_f(0)+ e_{f*}(0)|$=2m.Thus in this case we have $|(v_f(1)+ e_{f*}(1))-(v_f(0)+ e_{f*}(0))|\le 1$.

Case (ii): when m≤n and n=m+ m′
f(u)=1,

$f(u_i)=1$ for i=1, 2, ..., m-1,

and

$$f(v)=0,$$

$$f(v_i)=\begin{cases}0 & \text{for } i=1,2,\dots,m-1,\\ 1 & \text{for } i=m,\dots,n-1.\end{cases}$$

In this case we have $|v_f(1)+e_{f*}(1)|$ =2m +m'-1 and $|v_f(0)+e_{f*}(0)|$= 2m +m'. Thus in this case we have $|(v_f(1)+e_{f*}(1))-(v_f(0)+e_{f*}(0))|\le 1$.Hence, the double star graph $S_{m,n}$ is total vertex product cordial graph.

Example 2. The double star graph $S_{4,\,6}$ with its total vertex product cordial labeling is shown in Figure (vi).

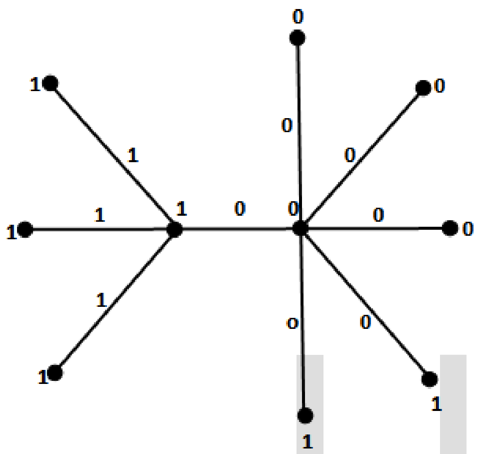


Figure (vi)

Theorem 3.3 The fan graph fn admits a total vertex product cordial labeling.

Proof. Let v be an apex vertex and $v_1, v_2, \dots, v_n$ be the other vertices of the fan graph f_n. To define f: V(G) →{ 0, 1 } and we consider the following cases:

case (i): when n is even

$$f(v)=1,$$

$$f(v_i)=\begin{cases}1 & \text{for } 1\le i\le \frac{n}{2},\\ 0 & \text{for } \left(\frac{n}{2}+1\right)\le i\le n.\end{cases}$$

In view of the above defined labeling pattern we have $|v_f(1)+e_{f*}(1)|=\frac{3n}{2}$and $|v_f(0)+e_{f*}(0)|=\frac{3n}{2}$. Thus in this case we have $|(v_f(1)+e_{f*}(1))-(v_f(0)+e_{f*}(0))|\le 1$.

case (ii): when n is odd

$$f(v)=0,$$

$$f(vi)=\begin{cases}0 & \text{for } 1\le i\le \left[\frac{n}{4}\right],\\ 1 & \text{for } \left[\frac{n}{4}\right]+1\le i\le n.\end{cases}$$

In view of the above defined labeling pattern we have $|v_f(1)+e_{f*}(1)|=\frac{3n+1}{2}$and $|v_f(0)+e_{f*}(0)|=\frac{3n-1}{2}$ .Thus in this case we have $|(v_f(1)+e_{f*}(1))-(v_f(0)+e_{f*}(0))|\le 1$.

Hence, the fan graph fn admits a total vertex product cordial labeling.

Example3. The fan graphs f_5 and f_6 with its total vertex product cordial labeling are shown in Figure (vii) and Figure (viii).

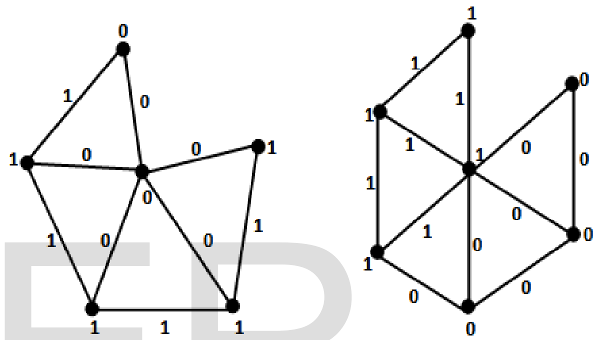


Figure (vii) and Figure (viii)

Theorem 3.4 The Helm graph H_n is total vertex product cordial graph, when n =3,4.

Proof. Let v be the apex vertex, $v_1, v_2, \dots, v_n$ be the vertices of degree 4 and $u_1, u_2, \dots, u_n$ be the pendant vertices of Hn. Then |V(Hn)|= 2n+1 and $|E(H_n)|$= 3n. We define f: V(G) →{ 0, 1 } we consider the following cases:

case (i): when n =3,4

$$f(v)=1,$$

$$f(v_i)=\begin{cases}1 & \text{for } 1\le i\le n-1,\\ 0 & \text{for } i=n.\end{cases}$$

and

$$f(v_i)=\begin{cases}1 & \text{for } j=1,\\ 0 & \text{for } 2\le j\le n\end{cases}$$

In this case we have $|v_f(1)+e_{f*}(1)$ = 3n-1 and $|v_f(0)+e_{f*}(0)|$= 2n+2, and we get

$|(v_f(1)+e_{f*}(1))-(v_f(0)+e_{f*}(0))| \leq 1$.

This completes the proof.

Example 4. The Helm graph H_3 and H_4 with total vertex product cordial graph.

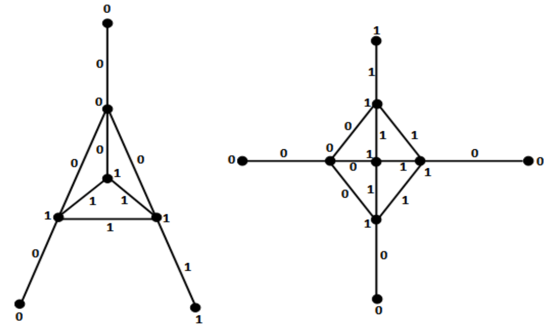


Figure (ix) and Figure (x)

Theorem 3.5 The Cycle graph C_n is total vertex product cordial graph.

Proof. Let $v_1, v_2, \ldots, v_n$ be the n vertices of cycle C_n. We define f: V(G) $\rightarrow$ { 0, 1 }, we consider the following cases:

case (i): when n is even and n>4.

$$f(v_i)=\begin{cases} 1 & \text{for } 1 \leq i \leq \frac{n}{2}, \\ 0 & \text{for } \frac{n}{2}+1 \leq i \leq n-2, \\ 1 & \text{for } i = n-1, \\ 0 & \text{for } i = n. \end{cases}$$

In this case we have $|v_f(1)+e_{f*}(1)| = \frac{n}{2}$ and also $|v_f(0)+e_{f*}(0)| = \frac{n}{2}$. Therefore $|(v_f(1)+e_{f*}(1))-(v_f(0)+e_{f*}(0))| \leq 1$.

case (ii): when n is odd

$$f(vi)=\begin{cases} 1 & \text{for } 1 \leq i \leq \left[\frac{n}{2}\right]+1, \\ 0 & \text{for } \left[\frac{n}{2}\right]+2 \leq i \leq n. \end{cases}$$

In this case also we have $|v_f(1)+e_{f*}(1)| = \frac{n}{2}$ and also $|v_f(0)+e_{f*}(0)| = \frac{n}{2}$.

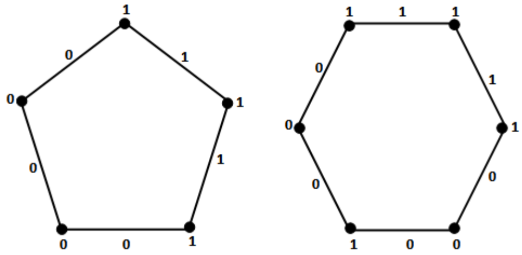
Therefore $|(v_f(1)+e_{f*}(1))-(v_f(0)+e_{f*}(0))| \leq 1$.

This completes the proof.

Remark 2 : C_n is not a total vertex cordial graph when n= 2,4.

We now give examples of C_6 and C_5 with total vertex product cordial labeling.

Example 5. The cycle C_6 and C_5 with total vertex product cordial labeling.



Figure(xi) **Figure(xii)**

Theorem 3.6 Every fully binary tree is total vertex product cordial labeling.

Proof. We know that every fully binary tree has odd number of vertices and even number of edges.

Let T be a fully binary tree and v be the root of T, v is called zero level vertex. If T has m levels with $v_1, v_2, \ldots, v_r \ldots$ left child and $u_1, u_2, \ldots, u_r \ldots$ right child. We define f: V(G) $\rightarrow$ { 0, 1} such that

$$f(v)=1,$$

$f(v_i)=1$ and and $f(u_i)= 0$ for all i.

Thus in this case we have $|(v_f(1)+e_{f*}(1))-(v_f(0)+e_{f*}(0))| \leq 1$. Hence, every fully binary tree is total vertex product cordial graph.

Example 6. In this example we see that every fully binary tree is total vertex product cordial graph.

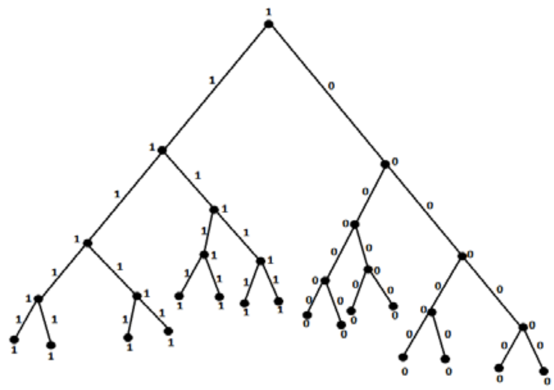


Figure (xiii)

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